

How to make the cooperation stable?

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Introduction

Cooperation is a basic form of human behavior. And for many practical reasons it is important that cooperation stable on a time interval under consideration. There are three important aspects which must be taken into account when the problem of stability of long-range cooperative agreements is investigated.

1. *Time-consistency (dynamic stability) of the cooperative agreements.* Time-consistency involves the property that, as the cooperation develops cooperating partners are guided by the same optimality principle at each instant of time and hence do not possess incentives to deviate from the previously adopted cooperative behavior.
2. *Strategic stability.* The agreement is to be developed in such a manner that at least individual deviations from the cooperation by each partner will not give any advantage to the deviator. This means that the outcome of cooperative agreement must be attained in some Nash equilibrium, which will guarantee the strategic support of the cooperation.
3. *Irrational behavior proofness.* This aspect must be also taken in account since not always one can be sure that the partners will behave rational on a long time interval for which the cooperative agreement is valid. The partners involved in the cooperation must be sure that even in the worst case scenario they will not loose compared with non cooperative behavior.

The mathematical tool based on payoff distribution procedure (PDP) or imputation distribution procedure (IDP) is developed to deal with the above mentioned aspects of cooperation.

1 Continuous time case

Consider n -person differential game $\Gamma(x_0, T - t_0)$ with prescribed duration and independent motions on the time interval $[t_0, T]$. Motion equations have the form:

$$\begin{aligned}\dot{x}_i &= f_i(x_i, u_i), \quad u_i \in U_i \subset R^\ell, x_i \in R^n, \\ x_i(t_0) &= x_i^0, \quad i = 1, \dots, n.\end{aligned}\tag{1}$$

It is assumed that the system of differential equations (1) satisfies all conditions necessary for the existence, prolongability and uniqueness of the solution for any n -tuple of measurable controls $u_1(t), \dots, u_n(t)$.

The payoff of player i is defined as:

$$H_i(x_0, T - t_0; u_1(\cdot), \dots, u_n(\cdot)) = \int_{t_0}^T h_i(x(\tau)) d\tau,$$

where $h_i(x)$ is a continuous function and $x(\tau) = \{x_1(\tau), \dots, x_n(\tau)\}$ is the solution of (1) when open-loop controls $u_1(t), \dots, u_n(t)$ are used and $x(t_0) = \{x_1(t_0), \dots, x_n(t_0)\} = \{x_1^0, \dots, x_n^0\}$.

Suppose that there exist an n -tuple of open-loop controls $\bar{u}(t) = (\bar{u}_1(t), \dots, \bar{u}_n(t))$ and the trajectory $\bar{x}(t)$, $t \in [t_0, T]$, such that

$$\begin{aligned}\max_{u_1(t), \dots, u_n(t)} \sum_{i=1}^n H_i(x_0, T - t_0; u_1(t), \dots, u_n(t)) &= \\ &= \sum_{i=1}^n H_i(x_0, T - t_0; \bar{u}_1(t), \dots, \bar{u}_n(t)) = \sum_{i=1}^n \int_{t_0}^T h_i(\bar{x}_i(\tau)) d\tau.\end{aligned}\tag{2}$$

The trajectory $\bar{x}(t) = (\bar{x}_1(t), \dots, \bar{x}_n(t))$ satisfying (2) we shall call "optimal cooperative trajectory".

Let $N = \{1, \dots, n\}$ be the set of players. Define in $\Gamma(x_0, T - t_0)$ characteristic function in a classical way:

$$\begin{aligned} V(x_0, T - t_0; N) &= \sum_{i=1}^n \int_{t_0}^T h_i(\bar{x}_i(\tau)) d\tau, \\ V(x_0, T - t_0; \emptyset) &= 0, \\ V(x_0, T - t_0; S) &= Val \Gamma_{S, N \setminus S}(x_0, T - t_0), \end{aligned} \tag{3}$$

where $Val \Gamma_{S, N \setminus S}(x_0, T - t_0)$ is a value of zero-sum game played between coalition S acting as first player and coalition $N \setminus S$ acting as player 2, with payoff of player S equal to:

$$\sum_{i \in S} H_i(x_0, T - t_0; u_1(\cdot), \dots, u_n(\cdot)).$$

Define $L(x_0, T - t_0)$ as imputation set in the game $\Gamma(x_0, T - t_0)$ (see Neumann and Morgenstern (1947)):

$$\begin{aligned} L(x_0, T - t_0) &= \{ \alpha = (\alpha_1, \dots, \alpha_n) : \\ &\alpha_i \geq V(x_0, T - t_0; \{i\}), \quad \sum_{i \in N} \alpha_i = V(x_0, T - t_0; N) \} . \end{aligned} \tag{4}$$

Regularized game $\Gamma_\alpha(x_0, T - t_0)$. For every $\alpha \in L(x_0, T - t_0)$ define the noncooperative game $\Gamma_\alpha(x_0, T - t_0)$, which differs from the game $\Gamma(x_0, T - t_0)$ only by payoffs defined along optimal cooperative trajectory $\bar{x}(\tau)$, $\tau \in [t_0, T]$.

Let $\alpha \in L(x_0, T - t_0)$. Define the imputation distribution procedure (IDP) (see Petrosjan (1993)) as function $\beta(\tau) = (\beta_1(\tau), \dots, \beta_n(\tau))$, $\tau \in [t_0, T]$ such that

$$\alpha_i = \int_{t_0}^T \beta_i(\tau) d\tau. \quad (5)$$

Denote by $H_i^\alpha(x_0, T - t_0; u_1(\cdot), \dots, u_n(\cdot))$ the payoff function in the game $\Gamma_\alpha(x_0, T - t_0)$ and by $x(\tau)$ the corresponding trajectory, then

$$H_i^\alpha(x_0, T - t_0; u_1(\cdot), \dots, u_n(\cdot)) = H_i(x_0, T - t_0; u_1(\cdot), \dots, u_n(\cdot))$$

if there does not exist such $t \in [t_0, T]$ that $x(\tau) = \bar{x}(\tau)$ for $\tau \in (t_0, t]$. Let $t = \sup\{t' : x(\tau) = \bar{x}(\tau), \tau \in [t_0, t']\}$ and $t > t_0$, then

$$\begin{aligned} H_i^\alpha(x_0, T - t_0; u_1(\cdot), \dots, u_n(\cdot)) &= \\ &= \int_{t_0}^t \beta_i(\tau) d\tau + H_i(\bar{x}(t), T - t; u_1(\cdot), \dots, u_n(\cdot)) = \\ &= \int_{t_0}^t \beta_i(\tau) d\tau + \int_t^T h_i(x(\tau)) d\tau. \end{aligned}$$

In a special case, when $x(\tau) = \bar{x}(\tau)$, $\tau \in [t_0, T]$ (if $x(\tau)$ is an optimal cooperative trajectory in the

sense of Eq. (2)), we have

$$H_i^\alpha(x_0, T - t_0; \bar{u}_1(\cdot), \dots, \bar{u}_n(\cdot)) = \int_{t_0}^T \beta_i(\tau) d\tau = \alpha_i.$$

By the definition of payoff function in the game $\Gamma_\alpha(x_0, T - t_0)$ we get that the payoffs along the optimal trajectory are equal to the components of the imputation $\alpha = (\alpha_1, \dots, \alpha_n)$.

Consider the current subgames (see Neumann and Morgenstern (1947)) — $\Gamma(\bar{x}(t), T - t)$ along $\bar{x}(t)$ and current imputation sets $L(\bar{x}(t), T - t)$. Let $\alpha(t) \in L(\bar{x}(t), T - t)$. Suppose that $\alpha(t)$ can be selected as differentiable function of t , $t \in [t_0, T]$.

Definition 1. The game $\Gamma_\alpha(x_0, T - t_0)$ is called regularization of the game $\Gamma(x_0, T - t_0)$ (α -regularization) if the IDP β is defined in such a way that

$$\alpha_i(t) = \int_t^T \beta_i(\tau) d\tau$$

or

$$\beta_i(t) = -\alpha'_i(t). \quad (6)$$

From (6) we get

$$\alpha_i = \int_{t_0}^t \beta_i(\tau) d\tau + \alpha_i(t), \quad (7)$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in L(x_0, T - t_0)$, and

$\alpha(t) = (\alpha_1(t), \alpha_2(t), \dots, \alpha_n(t)) \in L(\bar{x}(t), T - t)$. Suppose now that

$M(x_0, T - t_0) \subset L(x_0, T - t_0)$ is some optimality principle in the cooperative version of the game $\Gamma(x_0, T - t_0)$, and $M(\bar{x}(t), T - t) \subset L(\bar{x}(t), T - t)$ is the same optimality principle defined in the

subgames $\Gamma(\bar{x}(t), T - t)$ with initial coalitions on the optimal trajectory. M can be c -core, HM -solution, Shapley Value, Nucleous e.t.c. If $\alpha \in M(x_0, T - t_0)$, and $\alpha(t) \in M(\bar{x}(t), T - t)$ the condition (7) gives us the time consistence of the chosen imputation α , or the chosen optimality principle. Then we have the *time consistency (dynamic stability) of the chosen cooperative agreement*.

Consider now the problem of *strategic stability* of cooperative agreements. Based on imputation distribution procedure β , satisfying (5) we can prove the following basic theorem.

Theorem 1. In the regularization of the game $\Gamma_\alpha(x_0, T - t_0)$ for every $\varepsilon > 0$ there exist an ε -Nash equilibrium (Nash (1951)) with payoffs $\alpha = (\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$.

Proof. The proof is based on actual construction of the ε -Nash equilibrium in piecewise open-loop (POL) strategies with memory.

Remind the definition of POL strategies with memory in differential game. Denote by $\hat{x}(t)$ any admissible trajectory of the system (1) on the time interval $[t_0, t]$, $t \in [t_0, T]$.

The strategy $u_i(\cdot)$ of player i is called POL if it consists from the pair (a, σ) , where σ is a partition of time interval $[t_0, T]$, $t_0 < t_1 < \dots < t_l = T$ ($t_{k+1} - t_k = \delta > 0$), and a mapping a which corresponds to each point $(\hat{x}(t_k), t_k)$, $t_k \in \sigma$ an open-loop control $u_i(t)$, $t \in [t_k, t_{k+1})$.

Consider a family of associated with $\Gamma(x, T - t)$, but not with $\Gamma_\alpha(x, T - t)$ zero-sum games $\Gamma_{\{i\}, N \setminus \{i\}}(x, T - t)$ from the initial position x and duration $T - t$ between the coalition S consisting from a single player i and the coalition $N \setminus \{i\}$ with player's i payoff equal to

$$H_i(x, T - t; u_1(\cdot) \dots, u_n(\cdot)).$$

The payoff of player $N \setminus \{i\}$ in $\Gamma_{\{i\}, N \setminus \{i\}}(x, T - t)$ equals to $(-H_i)$. Let $\hat{u}(x, t; \cdot)$ be the ε -optimal POL strategy of player $N \setminus \{i\}$ in $\Gamma_{\{i\}, N \setminus \{i\}}(x, T - t)$. Note, that $\hat{u}(x, t; \cdot) = \{\hat{u}_j(x, t; \cdot)\}$, $j \in N \setminus \{i\}$.

Let $\hat{x}(t) = \{\hat{x}_1(t), \dots, \hat{x}_n(t)\}$ be the segment of an admissible trajectory of (1) defined on the time interval $[t_0, t]$, $t \in [t_0, T]$. For each $i \in \{1, \dots, n\}$ define $\bar{t}(i) = \sup\{t_i : \hat{x}_i(t_i) = \bar{x}_i(t_i)\}$ and $\bar{\bar{t}}(j) = \min_i \bar{t}(i) = \bar{t}(j)$. $\bar{\bar{t}}(j)$ lies in one of the intervals $[t_k, t_{k+1})$, $k = 0, 1, \dots, l-1$. Thus, $\bar{t}(i) - t_0$ is the length of the time interval starting from t_0 on which $x_i(t)$ coincides with $\bar{x}_i(t)$ — the i -th component of the cooperative trajectory $\bar{x}(t)$. And $\bar{\bar{t}}(j) - t_0$ is the length of the time interval starting from t_0 on which $x(t)$ coincides with cooperative trajectory $\bar{x}(t)$.

Define the following strategies of player $i \in N$.

$$u_i^*(\cdot) = \begin{cases} \bar{u}_i(t) & \text{for } (\hat{x}(t_k), t_k) \text{ on the optimal cooperative} \\ & \text{trajectory } \bar{x}(t) \text{ } (\hat{x}(\tau) = \bar{x}(\tau), \tau \in [t_0, t_k]); \\ \hat{u}_i(\hat{x}(t_{k+1}), t_{k+1}; \cdot) & i\text{-th component of the } \varepsilon/2\text{-optimal POL} \\ & \text{strategy of player } N \setminus \{j\} \text{ in the game} \\ \Gamma_{\{j\}, N \setminus \{j\}}(x(t_{k+1}), T - t_{k+1}), & \text{if } t_k \leq \bar{\bar{t}}(j) < t_{k+1}; \\ \text{arbitrary} & \text{for all other positions.} \end{cases}$$

Show that $u^*(\cdot) = (u_1^*(\cdot), \dots, u_n^*(\cdot))$ is ε -Nash equilibrium in $\Gamma_\alpha(x_0, T - t_0)$. The following equality holds

$$H_i(x_0, T - t_0; u^*(\cdot)) = H_i(x_0, T - t_0; u_1^*(\cdot), \dots, u_n^*(\cdot)) = \int_{t_0}^T \beta_i(t) dt = \alpha_i. \quad (8)$$

Consider the n -tuple $(u^*(\cdot) || u_i(\cdot))$ where player i changes his strategy $u_i^*(\cdot)$ on $u_i(\cdot)$.

We have to show that

$$H_i(x_0, T - t_0; u^*(\cdot)) \geq H_i(x_0, T - t_0; u^*(\cdot) || u_i(\cdot)) - \varepsilon. \quad (9)$$

for all $i \in N$ and all POL $u_i(\cdot)$ of player i .

It is easy to see that when the n -tuple $u^*(\cdot)$ is played the game develops along the optimal trajectory $\bar{x}(t)$. If in $(u^*(\cdot)||u_i(\cdot))$ the trajectory $\bar{x}(t)$ is also realized then (9) will be equality and thus true. Suppose now that in $(u^*(\cdot)||u_i(\cdot))$ the trajectory $x(t)$ different from $\bar{x}(t)$ is realized. Then let

$$\bar{t} = \inf\{t : \bar{x}(t) \neq x(t)\}.$$

and $\bar{t} \in [t_{k-1}, t_k)$. Since the motion of players are independent we get $\bar{x}_m(t_k) = x_m(t_k)$ for $m \in N \setminus \{i\}$ and $\bar{x}_i(t_k) \neq x_i(t_k)$ (but $\bar{x}_j(t_{k-1}) = x_j(t_{k-1})$ for $j \in N$). Then from the definition of $u^*(\cdot)$ it follows that the players $m \in N \setminus \{i\}$ will use their strategies $\hat{u}_m(\hat{x}(t_k), t_k; \cdot)$ which are $\frac{\varepsilon}{2}$ -optimal in a zero-sum game $\Gamma_{\{i\}, N \setminus \{i\}}(x(t_k), T - t_k)$ against the player i which deviates from the optimal trajectory on a time interval $[t_{k-1}, t_k)$.

If the players from the set $N \setminus \{i\}$ will use their strategies $\hat{u}_m(\hat{x}(t_k), t_k; \cdot)$, player i starting from position $x(t_k), t_k$ will get not more than

$$V(x(t_k), T - t_k; \{i\}) + \frac{\varepsilon}{2},$$

where $V(x(t_k), T - t_k; \{i\})$ is the value of the game $\Gamma_{\{i\}, N \setminus \{i\}}(x(t_k), T - t_k)$. Then the total payoff of player i in $\Gamma_\alpha(x_0, T - t_0)$ when the n -tuple of strategies $(u^*(\cdot)||u_i(\cdot))$ is played cannot exceed the amount

$$\int_{t_0}^{t_{k-1}} \beta_i(\tau) d\tau + V(x(t_k), t_k; \{i\}) + \frac{\varepsilon}{2} + \int_{t_{k-1}}^{t_k} h_i(x_i(\tau)) d\tau. \quad (10)$$

But the payoff of player i when the n -tuple $u^*(\cdot)$ is played is equal to

$$\alpha_i = \int_{t_0}^T \beta_i(\tau) d\tau = \int_{t_0}^{t_{k-1}} \beta_i(\tau) d\tau + \int_{t_{k-1}}^T \beta_i(\tau) d\tau = \int_{t_0}^{t_{k-1}} \beta_i(\tau) d\tau + \alpha_i(t_{k-1}). \quad (11)$$

By the definition of IDP (see (5), (6)), $\alpha_i(t_{k-1}) \in L(\bar{x}(t_{k-1}), T - t_{k-1})$,

$$\int_{t_{k-1}}^T \beta_i(\tau) d\tau = \alpha_i(t_{k-1}) \geq V(\bar{x}(t_{k-1}), T - t_{k-1}; \{i\}). \quad (12)$$

From the continuity of the function V and continuity of the trajectory $x(t)$ by appropriate choice of $\delta > 0$ ($t_{k+1} - t_k = \delta$) the following inequalities can be guaranteed:

$$|V(\bar{x}(t_{k-1}), T - t_{k-1}; \{i\}) - V(x(t_k), T - t_k; \{i\})| < \frac{\varepsilon}{4},$$

$$\int_{t_{k-1}}^T \beta_i(\tau) d\tau = \alpha_i(t_{k-1}) \geq V(x(t_k), T - t_k; \{i\}) - \frac{\varepsilon}{4}.$$

Compare $\alpha_i(t_{k-1})$ and $V(x(t_k), t_k; \{i\}) + \frac{\varepsilon}{2} + \int_{t_{k-1}}^{t_k} h_i(x_i(\tau)) d\tau$. By choosing $\delta = t_{k+1} - t_k$ sufficiently small one can achieve that the integral $\int_{t_{k-1}}^{t_k} h_i(x_i(\tau)) d\tau$ will be also small (less than $\varepsilon/4$).

Adding to both sides of (12) the amount $\int_{t_0}^{t_k-1} \beta_i(\tau) d\tau$ and using the previous inequality we get

$$\begin{aligned}
\alpha_i &= \int_{t_0}^{t_k-1} \beta_i(\tau) d\tau + \alpha_i(t_{k-1}) \geq \int_{t_0}^{t_k-1} \beta_i(\tau) d\tau + V(\bar{x}(t_{k-1}), T - t_{k-1}; \{i\}) \geq \\
&\geq \int_{t_0}^{t_k-1} \beta_i(\tau) d\tau + V(x(t_k), T - t_k; \{i\}) - \frac{\varepsilon}{4} \\
&\geq \int_{t_0}^{t_k-1} \beta_i(\tau) d\tau + V(x(t_k), T - t_k; \{i\}) - \frac{\varepsilon}{4} + \int_{t_{k-1}}^{t_k} h_i(\tau) d\tau - \frac{\varepsilon}{4} \\
&\geq \int_{t_0}^{t_k-1} \beta_i(\tau) d\tau + V(x(t_k), T - t_k; \{i\}) + \int_{t_{k-1}}^{t_k} h_i(\tau) d\tau - \frac{\varepsilon}{2} \\
&\geq \int_{t_0}^{t_k-1} \beta_i(\tau) d\tau + V(x(t_k), T - t_k; \{i\}) + \int_{t_{k-1}}^{t_k} h_i(\tau) d\tau + \\
&\quad + \frac{\varepsilon}{2} - \frac{\varepsilon}{2} - \frac{\varepsilon}{2}.
\end{aligned} \tag{13}$$

Here first four addends in the right part of the inequality constitute the upper bound of player i payoff when $(u^*(\cdot) || u_i^*(\cdot))$ is played. But α_i is the payoff of player i when $u^*(\cdot)$ is played, and we get

$$\begin{aligned}
H_i(x_0, T - t_0; u^*(\cdot)) &= \alpha_i \geq \\
&\geq \int_{t_0}^{t_k-1} \beta_i(\tau) d\tau + V(x(t_k), T - t_k; \{i\}) + \int_{t_{k-1}}^{t_k} h_i(\tau) d\tau + \frac{\varepsilon}{2} - \varepsilon \geq \\
&\geq H_i(x_0, T - t_0; u^*(\cdot) || u_i(\cdot)) - \varepsilon
\end{aligned} \tag{14}$$

and we get (9). The theorem is proved. \square

This means that the cooperative solution (any imputation) can be strategically supported in a regularized game $\Gamma_\alpha(x_0, T - t_0)$ (realized in a specially constructed Nash equilibrium) by the Nash equilibrium $u^*(\cdot)$ defined in the Theorem 1.

Conditions for the *irrational behavior proofness* of the cooperative solutions. Suppose now that in some intermediate instant of time the irrational behavior of some player (or players) will force the other players to leave the cooperative agreement, then the irrational behavior proofness condition (see D.W.K. Yeung (2007)) requires that the following inequality must be satisfied

$$V(x_0, T - t_0; \{i\}) \leq \int_{t_0}^t \beta_i(\tau) d\tau + V(\bar{x}(t), T - t; \{i\}), \quad i \in N. \quad (15)$$

If the IDP $\beta(t)$ can be chosen in such a way, that both time-consistency and irrational behavior proofness conditions are satisfied (the strategic stability as we have shown follows from time-consistency) the cooperative agreement about the choice of the imputation $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is stable.

From (15) we have the following condition for IDP $\beta(\tau) = (\beta_1(\tau), \beta_2(\tau), \dots, \beta_n(\tau))$:

$$\beta_i(\tau) \geq \frac{d}{d\tau} V(\bar{x}(\tau), T - \tau; \{i\}), \quad i = 1, \dots, n. \quad (16)$$

Not always all 3 conditions can be satisfied together. The next example show that in the discrete time case Shapley Value did not satisfy all these conditions.

2 Discrete time case

In what follows as basic model we shall consider the game in extensive form with perfect information.

Definition 2. A game tree is a finite oriented treelike graph K with the root x_0 .

We shall use the following notations. Let x be some vertex (position). We denote by $K(x)$ a subtree K with the root in x . We denote by $Z(x)$ immediate successors of x . The vertices y , directly following after x , are called alternatives in x ($y \in Z(x)$). The player who makes a decision in x (who selects the next alternative position in x), will be denoted by $i(x)$. The choice of player $i(x)$ in position x will be denoted by $\bar{x} \in Z(x)$.

Let $N = \{1, \dots, n\}$ — be the set of all players in the game.

Definition 3. A game in extensive form with perfect information (see Kuhn (1953)) $G(x_0)$ is a graph tree $K(x_0)$, with the following additional properties:

- The set of vertices (positions) is split up into $n + 1$ subsets P_1, P_2, \dots, P_{n+1} , which form a partition of the set of all vertices of the graph tree K . The vertices (positions) $x \in P_i$ are called players i personal positions, $i = 1, \dots, n$; vertices (positions) $x \in P_{n+1}$ are called terminal positions.
- In each final vertex (position) the system of real numbers $h(w) = (h_1(w), \dots, h_n(w))$, $w \in P_{n+1}$, $h_i(w) \geq 0$, $i = 1, \dots, n$ is defined. Where $h_i(w)$ is the payoff of player i in the final vertex (position).

Definition 4. A strategy of player i is a mapping $U_i(\cdot)$, which associate to each position $x \in P_i$ a unique alternative $y \in Z(x)$.

As in the previous case denote by $H_i(x; u_1(\cdot), \dots, u_n(\cdot))$ the payoff function of player $i \in N$ in

the subgame $G(x)$ starting from the position x .

$$H_i(x; u_1(\cdot), \dots, u_n(\cdot)) = h_i(x'_l)$$

where $x'_l \in P_{n+1}$ is the last vertex (position) in the path $x = (x'_1, x'_2, \dots, x'_l)$ realized in the subgame $G(x)$, when the n -tuple of strategies $(u_1(\cdot), \dots, u_n(\cdot))$ is played.

Denote by $\bar{u}(\cdot) = (\bar{u}_1(\cdot), \dots, \bar{u}_n(\cdot))$ the n -tuple of strategies and the trajectory (path) $\bar{x} = (\bar{x}_0, \bar{x}_1, \dots, \bar{x}_m)$, $\bar{x}_m \in P_{n+1}$ such that

$$\begin{aligned} \max_{u_1(\cdot), \dots, u_n(\cdot)} \sum_{i=1}^n H_i(x_0; u_1(\cdot), \dots, u_n(\cdot)) = \\ = \sum_{i=1}^n H_i(x_0; \bar{u}_1(\cdot), \dots, \bar{u}_n(\cdot)) = \sum_{i=1}^n h_i(\bar{x}_m). \end{aligned} \quad (17)$$

The path $\bar{x} = (\bar{x}_0, \dots, \bar{x}_m)$ satisfying Eq. (17) we shall call "optimal cooperative trajectory".

Define in $G(x_0)$ characteristic function in a classical way

$$\begin{aligned} V(x_0; N) &= \sum_{i=1}^n h_i(\bar{x}_m), \\ V(x_0; \emptyset) &= 0, \\ V(x_0; S) &= Val \Gamma_{S, N \setminus S}(x_0), \end{aligned}$$

where $Val \Gamma_{S, N \setminus S}(x_0)$ is a value of zero-sum game played between coalition S acting as first

player and coalition $N \setminus S$ acting as player 2, with payoff of player S equal to

$$\sum_{i \in S} H_i(x_0; u_1(\cdot), \dots, u_n(\cdot)).$$

Define $L(x_0)$ as imputation set in the game $G(x_0)$.

$$L(x_0) = \left\{ \alpha = (\alpha_1, \dots, \alpha_n) : \alpha_i \geq V(x_0; \{i\}), \quad \sum_{i \in N} \alpha_i = V(x_0; N) \right\}.$$

Regularized game $G_\alpha(x_0)$. For every $\alpha \in L(x_0)$ define the noncooperative game $G_\alpha(x_0)$, which differs from the game $G(x_0)$ only by payoffs defined along optimal cooperative path $\bar{x} = (\bar{x}_0, \dots, \bar{x}_m)$. Let $\alpha \in L(x_0)$. Define the imputation distribution procedure (IDP) as function $\beta_k = (\beta_1(k), \dots, \beta_n(k))$, $k = 0, 1, \dots, m$ such that

$$\alpha_i = \sum_{k=0}^m \beta_i(k). \quad (18)$$

Define by $H_i^\alpha(x_0; u_1(\cdot), \dots, u_n(\cdot))$ the payoff function in the game $G_\alpha(x_0)$ and by $\bar{x} = \{\bar{x}_0, \dots, \bar{x}_m\}$ the cooperative path

$$H_i^\alpha(x_0; u_1(\cdot), \dots, u_n(\cdot)) = H_i(x_0; u_1(\cdot), \dots, u_n(\cdot))$$

for all $u_1(\cdot), \dots, u_n(\cdot)$ such that the path $x = \{x_0, \dots, x_m\}$ differs from $\bar{x} = \{\bar{x}_0, \dots, \bar{x}_m\}$, and

$$H_i^\alpha(x_0; \bar{u}_1(\cdot), \dots, \bar{u}_n(\cdot)) = \alpha_i.$$

By the definition of the payoff function in the game $G_\alpha(x_0)$ we get that the payoffs along the optimal cooperative trajectory are equal to the components of the imputation $\alpha = (\alpha_1, \dots, \alpha_n)$.

Consider current subgames $G(\bar{x}_k)$ along the optimal path \bar{x} and current imputation sets $L(\bar{x}_k)$. Let $\alpha^k \in L(\bar{x}_k)$.

Definition 5. The game $G_\alpha(x_0)$ is called regularization of the game $G(x_0)$ (α -regularization) if the IDP β is defined in such a way that

$$\alpha_i^k = \sum_{j=k}^m \beta_i(j)$$

or $\beta_i(k) = \alpha_i^k - \alpha_i^{k+1}$, $i \in N$, $k = 0, 1, \dots, m-1$, $\beta_i(m) = \alpha_i^m$, $\alpha_i^0 = \alpha_i$.

Theorem 2. In the regularization of the game $G_\alpha(x_0)$ there exist a Nash equilibrium with payoffs $\alpha = (\alpha_1, \dots, \alpha_n)$.

Proof. Along the cooperative path we have

$$\alpha_i^k \geq V(\bar{x}_k; \{i\}), \quad i \in N, k = 0, 1, \dots, m.$$

since $\alpha^k = (\alpha_1^k, \dots, \alpha_n^k) \in L(\bar{x}_k)$ is an imputation in $G(\bar{x}_k)$ (note that here $V(\bar{x}_k; \{i\})$ is computed in the subgame $G(\bar{x}_k)$ but not $G_\alpha(\bar{x}_k)$). In the same time

$$\alpha_i^k = \sum_{j=k}^m \beta_i(j)$$

and we get

$$\sum_{j=k}^m \beta_i(j) \geq V(\bar{x}_k; \{i\}), \quad i \in N, k = 0, 1, \dots, m. \quad (19)$$

But $\sum_{j=k}^m \beta_i(j)$ is the payoff of player i in the subgame $G_\alpha(\bar{x}_k)$ along the cooperative path, and from (19) using the arguments similar to those in the proof of Theorem 1 one can construct the Nash equilibrium with payoffs $\alpha = (\alpha_1, \dots, \alpha_n)$ and resulting cooperative path $\bar{x} = (\bar{x}_0, \dots, \bar{x}_m)$.

Example. In this example as an imputation we shall consider Shapley value [Shapley (1953)]. Using the proposed regularization of the game we shall see that there exist a Nash equilibrium with payoffs equal to the components of the Shapley value.

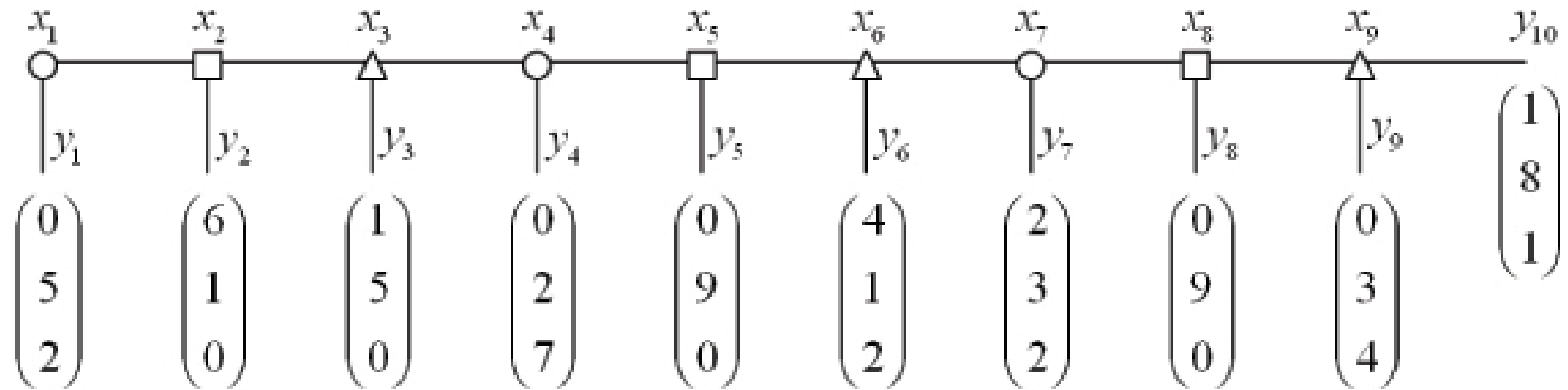


Fig. 1. Game $G(x_0)$

In the game $G(x_0)$, $N = \{1, 2, 3\}$, $P_1 = \{x_1, x_4, x_7\}$, $P_2 = \{x_2, x_5, x_8\}$, $P_3 = \{x_3, x_6, x_9\}$, $P_4 = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}$. $h(y_1) = (0, 5, 2)$, $h(y_2) = (6, 1, 0)$, $h(y_3) = (1, 5, 0)$, $h(y_4) = (0, 2, 7)$, $h(y_5) = (0, 9, 0)$, $h(y_6) = (4, 1, 2)$, $h(y_7) = (2, 3, 2)$, $h(y_8) = (0, 9, 0)$, $h(y_9) = (0, 3, 4)$, $h(y_{10}) = (1, 8, 1)$. The cooperative path is $\bar{x} = \{\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5, \bar{x}_6, \bar{x}_7, \bar{x}_8, \bar{x}_9, \bar{y}_{10}\}$.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_{10}
$V(x; \{1\})$	0	0	0	0	0	2	2	0	0	1
$V(x; \{2\})$	2	2	2	2	9	1	3	9	3	8
$V(x; \{3\})$	0	0	0	0	0	2	0	0	4	1
$V(x; \{1, 2\})$	7	7	6	9	9	5	9	9	3	9
$V(x; \{2, 3\})$	7	9	9	9	9	5	5	9	9	9
$V(x; \{1, 3\})$	6	6	6	7	0	6	4	0	4	2
$V(x; \{1, 2, 3\})$	10	10	10	10	10	10	10	10	10	10
$Sh(x; \{1\})$	$\frac{17}{6}$	$\frac{13}{6}$	$\frac{12}{6}$	$\frac{16}{6}$	$\frac{2}{6}$	$\frac{22}{6}$	$\frac{24}{6}$	$\frac{2}{6}$	$\frac{2}{6}$	1
$Sh(x; \{2\})$	$\frac{26}{6}$	$\frac{28}{6}$	$\frac{27}{6}$	$\frac{28}{6}$	$\frac{56}{6}$	$\frac{16}{6}$	$\frac{30}{6}$	$\frac{56}{6}$	$\frac{26}{6}$	8
$Sh(x; \{3\})$	$\frac{17}{6}$	$\frac{19}{6}$	$\frac{21}{6}$	$\frac{16}{6}$	$\frac{2}{6}$	$\frac{22}{6}$	$\frac{6}{6}$	$\frac{2}{6}$	$\frac{32}{6}$	1

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_{10}
$\beta_1(j)$	$\frac{2}{6}$	$\frac{4}{6}$	$\frac{1}{6}$	$\frac{14}{6}$	$-\frac{28}{6}$	$-\frac{2}{6}$	$\frac{22}{6}$	0	$-\frac{4}{6}$	1
$\beta_2(j)$	$-\frac{1}{6}$	$-\frac{2}{6}$	$\frac{1}{6}$	$-\frac{28}{6}$	$\frac{40}{6}$	$-\frac{14}{6}$	$-\frac{28}{6}$	$\frac{30}{6}$	$-\frac{22}{6}$	8
$\beta_3(j)$	$-\frac{1}{6}$	$-\frac{2}{6}$	$-\frac{2}{6}$	$\frac{14}{6}$	$-\frac{20}{6}$	$\frac{16}{6}$	$\frac{4}{6}$	$-\frac{30}{6}$	$\frac{26}{6}$	1

It can be easily seen that the inequality (19)

$$\sum_{j=k}^m \beta_i(j) \geq V(\bar{x}_k; \{i\})$$

for $i \in N$ holds in this case, but the irration-behavior-proofness condition is not satisfied.

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